Probeklausur — Kodierungstheorie 2015

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Versuchen Sie innerhalb der nächsten 50 Minuten die folgenden Aufgaben zu lösen. Sie können sich natürlich mit Ihren Nachbarn austauschen, sofern niemand wegen der Ruhestörung protestiert. Im Anschluss werden wir alle Aufgaben gemeinsam durchgehen, die korrekten Lösungen sehen und Gelegenheit haben, damit zusammenhängende Fragen zu diskutieren. Die Blätter mit Ihren Lösungen behalten Sie natürlich für sich.

1 Grundbegriffe

Complete the following statements.

1.1

A cyclic code of length n over a field F is

a linear code which contains with every codeword $c_1c_2\cdots c_n$ also the codeword $c_nc_1c_2\cdots c_{n-1}$.

1.2

The weight enumerator of the binary repetition code $\{0, 11\cdots 1\}$ of length n equals

$1 + x^n$.

1.3

The Reed-Solomon code $\mathrm{RS}_{11}(a, k)$ is defined as

$$\{(f(a_1), \cdots, f(a_n)) : f \in \mathbb{F}_{11}[x]_{<k}\},$$

where $a = (a_1, \cdots, a_n) \in \mathbb{F}_{11}$.

1.4

The projective 3-space over $\mathbb{F}_2$ is defined as

the set of all 1-dimensional subspaces of $\mathbb{F}_2^4$.

1.5

The number of cyclic codes of length 3 over $\mathbb{F}_2$ equals .

4 (since $x^3 - 1 = (x-1)(x^2+x+1)$ has exactly 4 monic divisors).

1.6

The number of zeros in F^r of a non-zero multivariate polynomial of degree d in $F[x_1, \dots, x_r]$is bounded to above by

$d \cdot |F|^{r-1}$ (by the Schwartz-Zippel lemma).

1.7

The control polynomial of the binary cyclic code of length 4 with generator polynomial $(x-1)^3$ equals .

$x - 1$ (since $x^4 - 1 = (x-1)^4$).

1.8

A field of order 9 is for example

$\mathbb{F}_3[x]/x^2 + 1$.

1.9

The number of linear maps $\mathbb{F}_3^3 \to \mathbb{F}_3^3$ which preserve the Hamming distance equals

$6 \cdot 8$ since the group of isometries equals the group of permutation matrices (here 6) times the group of invertible diagonal matrices (here 8).

1.10

The control matrix of the binary linear code $\{000, 111, 100, 011\}$ is the following matrix

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

2 Methoden

2.1

Let $P = \mathbb{P}^2(\mathbb{F}_3)$ be the projective plane over the field with three elements.

1. Compute the number of points of P.
2. Compute the number of lines in P.
3. How many points lie on a line?
4. How many lines go through a point.

(i) The number of points is $(3^3 - 1)/2 = 13$ (number of 3-vectors without zero divided by number of nonzero elements). (ii) The number of lines is the same as the number of points since every line $ax + by + cz = 0$ is given by a 3-vector modulo multiplication by a nonzero field element. (iii) 4 since these are the non-zero solutions of a linear equation $ax + by + cz = 0$ modulo multiplication by a non-zero field element. (iv) 4, which is the same answer as to (iii), and the proof is the same too (via duality).

2.2

Let C be the liner code generated by the "Two-out-of-five code" ToF. Compute the dimension, the minimal distance and the weight enumarator of C.

The vector $e := 11111^t$ is the only non-zero column vector such that $c \cdot e = 0$ (usual matrix product) for all c in ToF. It is then also the only non-zero vector such that $c \cdot e = 0$ for all c in C. Therefore, if A is a matrix whose rows form a basis of C, we have that e is the only non-zero solution of the system of linear equations $Ax = 0$. It follows that A has rank 4 (since number of unknowns 5 minus rank 4 equals dimension of solution space 1), i.e. that C has dimension 4, and that C it equals the space of solutions c of $c \cdot e = 0$ (since this solution space is of dimension 4 and contains C, hence equals C). In other words, C consists of all codewords with an even number of 1s. The minimal weight of C is then 2. The weight enumerator is $1 + 10x^2 + 4x^4$. Alternatively, one could argue that C has 16 or 32 elements, but since it is contained in the kernel of the map $w \mapsto \sum_i w_i$ it has then 16 elements, and is in fact equals to the kernel.