Solution to Blatt 8

1) a) $G = SL(2, \mathbb{F}_3)$, $\# G = 3.8$

- A 2-Sylow subgroup A has order 8 and every element in A has order dividing 8. But G contains exactly 8 elements of order dividing 8. So these form the unique 2-Sylow.
- # 3-Sylows subgroups divides 8 and it is $\equiv_3 1$, hence it equals 1 or 4. But "1" is impossible, since the 3-Sylow $\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$ is not normal. Hence, the 3-Sylow are $\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$ and two conjugates of this subgroup.

b) $G = SL(2, \mathbb{Z}/4\mathbb{Z})$, $\# G = 48 = 3.2^4$.

- # 3-Sylows divides 16 and $\equiv_3 1$, hence equals 1, 4, 16. The 3-Sylow $\langle (ST)^2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \rangle$ $\left(\begin{matrix} S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{matrix} \right)$ is not normal. Note that ~~two~~ different 3-Sylow have intersection $\{1\}$. Hence # elements of order 3 = 2 . # 3-Sylow. But we have

order	1	2	3	4	6
# elements	1	7	8	24	8

Hence, there are 4 3-Sylows, the 4 conjugates of $\langle (ST)^2 \rangle$.

- The # 2-Sylows divides 3 and it equals $\equiv_2 1$. Hence, there are 1 or 3 2-Sylows. But we have 32 elements whose order divides 16, hence there must be 3 2-Sylows.

2) • A is contained in a 2-Sylow subgroup. But A and the 2-Sylows has order 8. So, A equals to a 2-Sylow. Therefore, A is a subgroup of $G = SL(2, \mathbb{F}_3)$. From the second Sylow theorem, we know that all conjugates of A are also 2-Sylows. But by Exercise 1.a. we know that there is only one 2-Sylow, hence all conjugates of A equal to A, i.e. A is a normal subgroup of G.

• From Blatt 7, question 1, we know the class equation for G as:

$$\circledast\circledast \quad 24 = \underbrace{1}_{\text{order } 1} + \underbrace{1}_{\text{order } 2} + \underbrace{6}_{\text{order } 4} + \overbrace{\underbrace{4+4}_{\text{order } 3} + \underbrace{4+4}_{\text{order } 6}}^{\circledast}$$

Every normal subgroup has order which is the sum of the terms on the RHS of $\circledast\circledast$, and ~~the order~~ divides 24, i.e. it equals 1, 2, 3, 4, 6, 8, 12.

- If $-1 \notin N$, then

  $|N| = 1 + \text{even}$, i.e. $|N| = 1$ ✓

  or $|N| = 3$ ✗. (since there are 4 3-Sylows)

- If $-1 \in N$, then

  $|N| = 2 + $ some of ~~th~~ terms from $\circledast$, i.e.

  $= 2 + 0$ or ✓

  $= 2 + 6$ or ✓ (2-Sylow)

  1) $= 2 + 4$ or No!

  2) $= 2 + 6 + 4$ or No!

  ~~2~~ $2 + \text{all} = 24$ ✓

1) Since then N contains a 3-Sylow, hence all 4 3-Sylows, hence all elements of order 3, i.e. 8 ✗.

2) Since then N contains all elts of order 3, i.e. 8 elements and

it also contains an element of order 4, hence all elements of order 4, i.e. 6 elements. So, N would have at least 14 elements $\lightning$.

Therefore the ~~only~~ normal subgroups of G are $\{1\}$, $\{\pm 1\}$, A and G.

Let $\chi : G \longrightarrow \mathbb{C}^*$ a group homomorphism.

Let $K = \ker(\chi)$. Note $G/K \hookrightarrow \mathbb{C}^*$. Then

$K = \{1\}$ $\lightning$. since G is not abelian,

$K = \{\pm 1\}$ " "

$K = G$ ok, $\chi \equiv 1$

$K = A$:

G/A has order 3 generated by TA, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then χ ~~is one~~ equals :

$$\begin{aligned} TA &\longmapsto \rho \\ T^2A &\longmapsto \rho^2 \\ A &\longmapsto 1 \end{aligned} \quad \text{or} \quad \begin{aligned} TA &\longmapsto \rho^2 \\ T^2A &\longmapsto \rho \\ A &\longmapsto 1 . \end{aligned}$$

Here ρ is a primitive 3^{rd} root of unity.

4) • $|G| = pq$, say $p > q$.

$\#$ p-Sylow $= 1, q$ and it is congruent to 1 modulo p. Then $\#$ p-Sylow $\neq q$ since $q < p$ and then $q \not\equiv_p 1$. Hence, there is exactly one p-Sylow, which is hence normal.

• $|G| = pq^2$

$\#$ p-Sylows $1, q, q^2$ and $\equiv_p 1$

$\#$ q-Sylows $1, p$ and $\equiv_q 1$

If one of the above numbers is 1, then the group is not simple. So, assume that both #'s are $\neq 1$. Then $p \equiv_q 1$, hence $p > q$. But then also $p \nmid q-1$, i.e. $q \not\equiv_p 1$. Hence we have that # p-Sylow $= q^2 \equiv_p 1$ and # q-Sylow $= p \equiv_q 1$. But $p \mid q^2 - 1 = (q-1)(q+1)$ implies $p \mid q-1$ or $p \mid q+1$. But $p \mid q-1$ is false. Therefore $p \mid q+1$. But since also $p \equiv_q 1$ we have therefore $p = q+1$. This is only possible when $q = 2$ and $p = 3$, i.e. when $|G| = 12$. But we know that a group of order 12 is not simple.