Blatt 12 – Übung 4

We denote $R := \mathbb{Z}[x]/(2,x)$. We have that $R \simeq \mathbb{F}_2$, since the epimorphism $\mathbb{Z}[x] \longrightarrow \mathbb{F}_2$, $\sum a_i x^i \longmapsto a_0 + 2\mathbb{Z}$ has kernel $(2,x)$. We have that $V := I/(2,x)I$ is a vector space over R:

Scalar multiplication: $(a+(2,x),\, p+(2,x)I) \longmapsto ap+(2,x)I$.

The well-definedness of the scalar multiplication:

If $a \equiv a' \bmod (2,x)$, then $(a-a')p \in (2,x)I$, since $p \in I$.

If $p \equiv p' \bmod (2,x)I$, then $a(p-p') \in (2,x)I$.

- If $I = (f_1, \dots f_r)$, then $I = \mathbb{Z}[x]f_1 + \dots + \mathbb{Z}[x]f_r$,

hence, $V = R\bar{f_1} + \dots + R\bar{f_r}$, where $\bar{f_i} = f_i + (2,x)I$.

Hence, $r \geq \dim V$.

- The elements $\overline{2^{n-1}}, \overline{2^{n-2}x}, \dots \overline{x^{n-1}}$ are lin. ind. over R:

($\Rightarrow r \geq \dim V = n$, i.e. I can never have less than r gens)

Namely, let $a_1\overline{2^{n-1}} + \dots + a_n\overline{x^{n-1}} = 0$ in V, where $a_i = \varepsilon_i + (2,x) \in R$. w.l.o.g. we can assume that $\varepsilon_i = 0,1 \in \mathbb{Z}$, i.e. we have

$$h := \varepsilon_1 2^{n-1} + \varepsilon_2 2^{n-2}x + \dots + \varepsilon_n x^{n-1} \in (2,x)I$$

$$(2,x)I = (2,x)(2^{n-1}, 2^{n-2}x, \dots x^{n-1}) = (2^n, 2^{n-1}x, \dots x^n)$$

Setting $X=0$, we obtain $h(0)=\varepsilon_1 2^{n-1} \in 2^n\mathbb{Z} \overset{\circledast}{=} (2,x)I(0)$.

Here, actually, for $\circledast$ we apply the ring homomorphism $\mathbb{Z}[X] \longrightarrow \mathbb{Z}$, $p \longmapsto p(0)$. Hence, we obtain $\varepsilon_1=0$.

We then have $h=xh_1 \subseteq (2,x)I$ (here $h_1=\varepsilon_2 2^{n-2}+\dots+\varepsilon_n x^{n-2}$).

Then, $h=xh_1 \in (2,x)I = 2^n\mathbb{Z}[X]+xI_1$ (here $I_1=(2^{n-1},\dots,x^{n-1})$).

This implies that $h_1 \in 2^n \frac{p}{x}+p'$, where $p\in\mathbb{Z}[X]$, $p'\in I_1$. But since $2^n\in I_1$, ~~we~~ we have $2^n\frac{p}{x}\in I_1$, and therefore, $h_1\in I_1$. Similar to the previous case, we obtain, $\varepsilon_2=0$. Continuing in this way we obtain that all $\varepsilon_i=0$, i.e. the elements $\overline{2^{n-1}},\dots,\overline{x^{n-1}}$ are lin. ind. over R.